

A Characterization of the Two-weight Inequality for Riesz Potentials on Cones of Radially Decreasing Functions

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Abstract

We establish necessary and sufficient conditions on a weight pair (v, w) governing the boundedness of the Riesz potential operator I_α defined on a homogeneous group G from $L^p_{dec,r}(w, G)$ to $L^q(v, G)$, where $L^p_{dec,r}(w, G)$ is the Lebesgue space defined for non-negative radially decreasing functions on G . The same problem is also studied for the potential operator with product kernels I_{α_1, α_2} defined on a product of two homogeneous groups $G_1 \times G_2$. In the latter case weights, in general, are not of product type. The derived results are new even for Euclidean spaces. To get the main results we use Sawyer type duality theorems (which are also discussed in this paper) and two-weight Hardy type inequalities on G and $G_1 \times G_2$ respectively.

Keywords: Riesz potential, multiple Riesz potential, homogeneous group, cone of decreasing functions, two-weight inequality, Sawyer's duality theorem

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1. Introduction

A homogeneous group is a simply connected nilpotent Lie group G on a Lie algebra \mathfrak{g} with the one-parameter group of transformations $\delta_t = \exp(A \log t)$, $t > 0$, where A is a diagonalized linear operator in G with positive eigenvalues. In the homogeneous group G the mappings $\exp \circ \delta_t \circ \exp^{-1}$, $t > 0$, are automorphisms in G , which will be again denoted by δ_t . The number $Q = \text{tr } A$ is the homogeneous dimension of G . The symbol e will stand for the neutral element in G .

It is possible to equip G with a homogeneous norm $r : G \rightarrow [0, \infty)$ which is continuous on G , smooth on $G \setminus \{e\}$ and satisfies the conditions:

- (i) $r(x) = r(x^{-1})$ for every $x \in G$;
- (ii) $r(\delta_t x) = tr(x)$ for every $x \in G$ and $t > 0$;
- (iii) $r(x) = 0$ if and only if $x = e$;
- (iv) There exists $c_0 > 0$ such that

$$r(xy) \leq c_0(r(x) + r(y)), \quad x, y \in G.$$

In the sequel we denote by $B(a, t)$ an open ball with the center a and radius $t > 0$, i.e.

$$B(a, t) := \{y \in G; r(ay^{-1}) < t\}.$$

It can be observed that $\delta_t B(e, 1) = B(e, t)$.

Let us fix a Haar measure $|\cdot|$ in G such that $|B(e, 1)| = 1$. Then $|\delta_t E| = t^Q |E|$. In particular, $|B(x, t)| = t^Q$ for $x \in G$, $t > 0$.

Examples of homogeneous groups are: the Euclidean n -dimensional space \mathbb{R}^n , the Heisenberg group, upper triangular groups, etc. For the definition and basic properties of the homogeneous group we refer to [9], p. 12.

An everywhere positive function ρ on G will be called a weight. Denote by $L^p(\rho, G)$ ($1 < p < \infty$) the weighted Lebesgue space, which is the space of all measurable functions $f : G \rightarrow \mathbb{C}$ defined by the norm

$$\|f\|_{L^p(\rho, G)} = \left(\int_G |f(x)|^p \rho(x) dx \right)^{\frac{1}{p}} < \infty.$$

If $\rho \equiv 1$, then we use the notation $L^p(G)$.

Denote by $\mathcal{DR}(G)$ the class of all radially decreasing functions on G with values in \mathbb{R}_+ , i.e. the fact that $\phi \in \mathcal{DR}(G)$ means that there is decreasing $\tilde{\phi} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\phi(x) = \tilde{\phi}(r(x))$. In the sequel we will use the symbol ϕ itself for $\tilde{\phi}$; the fact that $\phi \in \mathcal{DR}(G)$ will be written also by the symbol $\phi \downarrow r$. Let G_1 and G_2 be homogeneous groups. We say that a function $\psi : G_1 \times G_2 \mapsto \mathbb{R}_+$ is radially decreasing if it is such in each variable separately uniformly to another one. The fact that ψ is radially decreasing on $G_1 \times G_2$ will be denoted as $\psi \in \mathcal{DR}(G_1 \times G_2)$.

Let

$$(I_\alpha f)(x) = \int_G f(y) (r(xy^{-1}))^{\alpha-Q} dy, \quad 0 < \alpha < Q,$$

be the Riesz potential defined on G , where r is the homogeneous norm and dy is the normalized Haar measure on G . The operator I_α plays a fundamental role in harmonic analysis, e.g., in the theory of Sobolev embeddings, in the theory of sublaplacians on nilpotent groups etc. Weighted estimates for multiple Riesz potentials can be applied, for example, to establish Sobolev and Poincaré inequalities on product spaces (see, e.g., [20]).

Let G_1 and G_2 be homogeneous groups with homogeneous norms r_1 and r_2 and homogeneous dimensions Q_1 and Q_2 respectively. We define the potential operator on $G_1 \times G_2$ as follows

$$I_{\alpha, \beta} f(x, y) = \iint_{G_1 \times G_2} f(t, \tau) (r_1(xt^{-1}))^{\alpha-Q_1} (r_2(y\tau^{-1}))^{\beta-Q_2} dt d\tau, \quad (x, y) \in G_1 \times G_2, \quad 0 < \alpha < Q_1, \quad 0 < \beta < Q_2.$$

Our aim is to derive two-weight criteria for I_α on the cone of radially decreasing functions on G . The same problem is also studied for the potential operator with product kernels $I_{\alpha, \beta}$ defined on a product of two homogeneous groups, where only the right-hand side weight is of product type. As far as we know the derived results for $I_{\alpha, \beta}$ are new even in the case of Euclidean spaces. The proofs of the main results are based on E. Sawyer (see [18]) type duality theorem

which is also true for homogeneous groups (see Propositions C and E below) and Hardy type two-weight inequalities in homogeneous groups. Analogous results for multiple potential operators defined on \mathbb{R}_+^n with respect to the cone of non-negative decreasing functions on \mathbb{R}_+^n were studied in [16], [15]. It should be emphasized that the two-weight problem for multiple Hardy operator for the cone of decreasing functions on \mathbb{R}_+^n was investigated by S. Barza, H. P. Heinig and L. -E. Persson [4] under the restriction that both weights are of product type.

Historically the one-weight inequality for the classical Hardy operator was characterized by M. A. Arino and B. Muckenhoupt [3] under the so called B_p condition. The same problem for multiple Hardy transform was studied by N. Arcozzi, S. Barza, J. L. Garcia-Domingo and J. Soria [2]. This problem in the two-weight setting was solved by E. Sawyer [18]. Some sufficient conditions guaranteeing the two-weight inequality for the Riesz potential I_α on \mathbb{R}^n was given by Y. Rakotondratsimba [17]. In particular, the author showed that I_α is bounded from $L_{dec,r}^p(w, \mathbb{R}^n)$ to $L^q(v, \mathbb{R}^n)$ if the weighted Hardy operators $(\mathcal{H}f)(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y|<|x|} f(y)dy$ and $(\mathcal{H}'f)(x) = \int_{|y|>|x|} \frac{f(y)}{|y|^{n-\alpha}} dy$ are bounded from $L^p(w, \mathbb{R}^n)$ to $L^q(v, \mathbb{R}^n)$. In fact the author studied the problem on the cone of monotone decreasing functions.

Now we give some comments regarding the notation: in the sequel under the symbol $A \approx B$ we mean that there are positive constants c_1 and c_2 (depending on appropriate parameters) such that $c_1 A \leq B \leq c_2 A$; $A \ll B$ means that there is a positive constant c such that $A \leq cB$; integral over a product set $E_1 \times E_2$ from g will be denoted by $\iint_{E_1 \times E_2} g(x, y) dx dy$

or $\int_{E_1} \int_{E_2} g(x, y) dx dy$; for a weight functions w and w_i on G , by the symbols $W(t)$ and $W_i(t)$ will be denoted the integrals $\int_{B(e, t)} w(x) dx$ and $\int_{B(e_i, t)} w_i(x) dx$ respectively; for a weight w on $G_1 \times G_2$, we denote $W(t, \tau) := \int_{B(e_1, t) \times B(e_2, \tau)} w(x, y) dx dy$, where e_1 and e_2 are neutral elements in G_1 and G_2 respectively. Finally we mention that constants (often different constants in one and the same lines of inequalities) will be denoted by c or C . The symbol p' stands for the conjugate number of p : $p' = p/(p-1)$, where $1 < p < \infty$.

2. Preliminaries

We begin this section with the statements regarding polar coordinates in G (see e.g., [9], P. 14).

Proposition A. *Let G be a homogeneous group and let $S = \{x \in G : r(x) = 1\}$. There is a (unique) Radon measure σ on S such that for all $u \in L^1(G)$,*

$$\int_G u(x) dx = \int_0^\infty \int_S u(\delta_t \bar{y}) t^{Q-1} d\sigma(\bar{y}) dt.$$

Let a be a positive number. The two-weight inequality for the Hardy-type transforms

$$(H^a f)(x) = \int_{B(e, ar(x))} f(y) dy, \quad x \in G,$$

$$(\tilde{H}^a f)(x) = \int_{G \setminus B(e, ar(x))} f(y) dy, \quad x \in G,$$

reads as follows (see [8], Ch.1 for more general case, in particular for quasi-metric measure spaces):

Theorem A. *Let $1 < p \leq q < \infty$ and let a be a positive number. Then*

(i)

The operator H^a is bounded from $L^p(u_1, G)$ to $L^q(u_2, G)$ if and only if

$$\sup_{t>0} \left(\int_{G \setminus B(e, t)} u_2(x) dx \right)^{1/q} \left(\int_{B(e, at)} u_1^{1-p'}(x) dx \right)^{1/p'} < \infty.$$

(ii)

The operator \tilde{H}^a is bounded from $L^p(u_1, G)$ to $L^q(u_2, G)$ if and only if

$$\sup_{t>0} \left(\int_{B(e, t)} u_2(x) dx \right)^{1/q} \left(\int_{G \setminus B(e, at)} u_1^{1-p'}(x) dx \right)^{1/p'} < \infty.$$

We refer also to [7] for the Hardy inequality written for balls with center at the origin.

In the sequel we denote H^1 by H .

The following statement for Euclidean spaces was derived by S. Barza, M. Johansson and L. -E. Persson [5].

Proposition B. *Let w be a weight function on G and let $1 < p < \infty$. If $f \in \mathcal{DR}(G)$ and $g \geq 0$, then*

$$\sup_{f \downarrow r} \frac{\int_G f(x)g(x)dx}{\left(\int_G f(x)^p w(x)dx\right)^{1/p}} \approx \|w\|_{L^1(G)}^{-1/p} \|g\|_{L^1(G)} + \left(\int_G H^{p'}(r(x))W^{-p'}(r(x))w(x)dx\right)^{1/p'},$$

$$\text{where } H(t) = \int_{B(e,t)} g(x)dx, \quad W(t) = \int_{B(e,t)} w(x)dx.$$

The proof of Proposition B repeats the arguments (for \mathbb{R}^n) used in the proof of Theorem 3.1 of [5] taking Proposition A and the following lemma into account.

Lemma A. *let $1 < p < \infty$. For a weight function w , the inequality*

$$\int_G w(x) \left(\int_{G \setminus B(e,r(x))} f(y)dy \right)^p dx \leq p \int_G f^p(x) W^p(r(x)) w^{1-p}(x) dx, \quad f \geq 0,$$

holds.

Proof of this lemma is based on Theorem A (part (ii)) taking $a = 1$, $p = q$, $u_2(x) = v(x)$, $u_1 = w^{1-p}(x)W^p(r(x))$ there. Details are omitted. \square

Corollary A. *Let the conditions of Proposition B be satisfied and let $\int_G w(x)dx = \infty$. Then the following relation holds:*

$$\sup_{f \downarrow r} \frac{\int_G f(x)g(x)dx}{\left(\int_G f^p(x)w(x)dx\right)^{1/p}} \approx \left(\int_G H^{p'}(r(x))W(r(x))w(x)dx\right)^{1/p}.$$

Corollary A implies the following duality result which follows by the standard way (see [18], [5] for details).

Proposition C. *Let $1 < p, q < \infty$ and let v, w be weight functions on G with $\int_G w(x)dx = \infty$. Then the integral operator T defined on functions on G is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$ if and only if*

$$\left(\int_G \left(\int_{B(e,r(x))} (T^*g)(y)dy \right)^{p'} W^{-p'}(r(x))w(x)dx \right)^{1/p'} \leq C \left(\int_G g^{q'}(x)v^{1-q'}(x)dx \right)^{1/q'} \quad (2.1)$$

holds for every positive measurable g on G .

The next statement yields the criteria for the two-weight boundedness of the operator H on the cone $\mathcal{DR}(G)$. In particular the following statement is true:

Theorem B. *Let $1 < p \leq q < \infty$ and let v and w be weights on G such that $\|w\|_{L^1(G)} = \infty$. Then H is bounded from $L_{dec,r}^p(w, G)$ to $L_v^q(v, G)$ if and only if*

$$(i) \quad \sup_{t>0} \left(\int_{B(e,t)} w(x)dx \right)^{-1/p} \left(\int_{B(e,t)} v(x)r^{Qq}(x)dx \right)^{1/q} < \infty;$$

$$(ii) \quad \sup_{t>0} \left(\int_{B(e,t)} r^{Qp'}(x)W^{-p'}(r(x))w(x)dx \right)^{1/p'} \left(\int_{G \setminus B(e,t)} v(x)dx \right)^{1/q} < \infty.$$

Proof of this statement follows by the standard way applying Proposition C (see e.g. [18], [5]). \square .

Definition 2.1. Let ρ be a locally integrable a.e. positive function on G . We say that ρ satisfies the doubling condition at e ($\rho \in DC(G)$) if there is a positive constant $b > 1$ such that for all $t > 0$ the following inequality holds:

$$\int_{B(e,2t)} \rho(x)dx \leq b \int_{B(e,t)} \rho(x)dx.$$

Further, we say that $w \in DC^{\gamma,p}(G)$, where $1 < p < \infty$, $0 < \gamma < Q/p$, if there is a positive constant b such that for all $t > 0$

$$\int_{G \setminus B(e,t)} r^{\gamma p'}(x) W^{-p'}(r(x)) w(x) dx \leq b \int_{G \setminus B(e,2t)} r^{\gamma p'}(x) W^{-p'}(r(x)) w(x) dx.$$

Remark 2.1. It is also to check that under the assumption $1 < p < \infty$, $0 < \gamma < Q/p$ the condition $w \in DC^{\gamma,p}(G)$ is satisfied for $w \equiv \text{const}$.

Definition 2.2. We say that a locally integrable a.e. positive function ρ on $G_1 \times G_2$ satisfies the doubling condition with respect to the second variable ($\rho \in DC^{(s)}(y)$) uniformly to the first one if there is a positive constant c such that for all $t > 0$ and almost every $x \in G_1$ the following inequality holds:

$$\int_{B(e_2,2t)} \rho(x,y)dy \leq c \int_{B(e_2,t)} \rho(x,y)dy.$$

Analogously is defined the class of weights $DC^{(s)}(x)$.

3. Riesz Potentials on G

The main result of this section reads as follows:

Theorem 3.1. Let $1 < p \leq q < \infty$ and let v and w be weights such that either $w \in DC^{\alpha,p}(G)$ or $v \in DC(G)$; let $\|w\|_{L^1(G)} = \infty$. Then the operator I_α is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$ if and only if

(i)

$$\sup_{t>0} \left(\int_{B(e,t)} w(x)dx \right)^{-1/p} \left(\int_{B(e,t)} r^{\alpha q}(x) v(x)dx \right)^{1/q} < \infty; \quad (3.1)$$

(ii)

$$\sup_{t>0} \left(\int_{B(e,t)} r^{p'Q}(x) W^{-p'}(r(x)) w(x)dx \right)^{1/p'} \left(\int_{G \setminus B(e,t)} r^{(\alpha-Q)q}(x) v(x)dx \right)^{1/q} < \infty; \quad (3.2)$$

(iii)

$$\sup_{t>0} \left(\int_{B(e,t)} v(x)dx \right)^{1/q} \left(\int_{G \setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}(r(x)) w(x)dx \right)^{1/p'} < \infty. \quad (3.3)$$

To prove this result we need to prove some auxiliary statements.

Lemma 3.1. Let $0 < \alpha < Q$ and let c_o be the constant from the triangle inequality of r . Then there is a positive constant c depending only on Q , α and c_o such that for all $s \in B(e, r(x)/2)$,

$$I(x, y) := \int_{B(e, r(x)) \setminus B(e, 2c_o r(y))} r(ty^{-1})^{\alpha-Q} dt \leq cr(xy^{-1})^\alpha. \quad (3.4)$$

Proof. We have

$$I(x, y) = \int_0^\infty |\{t \in G : r(ty^{-1})^{\alpha-Q} > \lambda\} \cap B(e, r(x)) \setminus B(e, 2c_o r(y))| d\lambda = \int_0^{r(xy^{-1})^{\alpha-Q}} (\dots) + \int_{r(xy^{-1})^{\alpha-Q}}^\infty (\dots) =: I^{(1)}(x, y) + I^{(2)}(x, y).$$

Observe that, by the triangle inequality for r , we have $r^Q(x) \leq c_0^Q 2^{Q-1}(r^Q(xy^{-1}) + r^Q(y))$. This implies that $r^Q(x) - (2c_0)^Q r^Q(y) \leq c_0^Q 2^{Q-1} r^Q(xy^{-1})$. Hence,

$$I^{(1)}(x, y) \leq r(xy^{-1})^{\alpha-Q} |B(e, r(x)) \setminus B(e, 2c_0 r(y))| = r(xy^{-1})^{\alpha-Q} (r^Q(x) - (2c_0)^Q r^Q(y)) \leq cr(xy^{-1})^\alpha.$$

Further, it is easy to see that

$$I^{(2)}(x, y) \leq cr(xy^{-1})^\alpha.$$

Finally we have (3.4). □

Let us introduce the following potential operators

$$(J_\alpha f)(x) = \int_{B(e, 2c_0 r(x))} f(y) r^{\alpha-Q}(xy^{-1}) dy, \quad (S_\alpha f)(x) = \int_{G \setminus B(e, 2c_0 r(x))} f(y) r^{\alpha-Q}(xy^{-1}) dy, \quad x \in G, \quad 0 < \alpha < Q.$$

It is easy to see that

$$I_\alpha f = J_\alpha f + S_\alpha f. \quad (3.5)$$

We need also to introduce the following weighted Hardy operator

$$(H_\alpha f)(x) = r(x)^{\alpha-Q} (Hf)(x).$$

Proposition 3.1. *The following relation holds for all $f \in \mathcal{DR}(G)$*

$$J_\alpha f \approx H_\alpha f. \quad (3.6)$$

Proof. We have

$$(J_\alpha f)(x) = \int_{B(e, r(x)/2c_0)} f(y) r^{\alpha-Q}(xy^{-1}) dy + \int_{B(e, 2c_0 r(x)) \setminus B(e, r(x)/(2c_0))} f(y) r^{\alpha-Q}(xy^{-1}) dy =: (J_\alpha^{(1)} f)(x) + (J_\alpha^{(2)} f)(x).$$

If $y \in B(e, r(x)/2c_0)$, then $r(x) \leq c_0(r(xy^{-1}) + r(y)) \leq c_0 r(xy^{-1}) + r(x)/2$. Hence $r(x) \leq 2c_0(r(xy^{-1}))$. Consequently,

$$(J_\alpha^{(1)} f)(x) \leq c(H_\alpha f)(x).$$

Applying now the fact that $f \in \mathcal{DR}(G)$ we see that

$$(J_\alpha^{(2)} f)(x) \leq \int_{B(e, r(x)/2c_0) \setminus B(e, 2c_0 r(x))} f(r(x)/2c_0) r^{\alpha-Q}(xy^{-1}) dy \leq cf(r(x)/2c_0) r(x)^\alpha \leq c(H_\alpha f)(x).$$

□

Lemma 3.2. *Let $1 < p \leq q < \infty$ and let v and w be weights on G such that $\|w\|_{L^1(G)} = \infty$. Then the operator S_α is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$ if*

$$\sup_{t>0} \left(\int_{G \setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}(r(x)) w(x) dx \right)^{1/p'} \left(\int_{B(e,t/(2c_0))} v(x) dx \right)^{1/q} < \infty.$$

Conversely, if S_α is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$, then the condition

$$\sup_{t>0} \left(\int_{G \setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}(x) w(x) dx \right)^{1/q} \left(\int_{B(e,t/(4c_0))} v(x) dx \right)^{1/p'} < \infty$$

is satisfied. Furthermore, if either $w \in DC^{\alpha,p}$ or $v \in DC(G)$, then the operator S_α is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$ if and only if

$$\sup_{t>0} \left(\int_{G \setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}(r(x)) w(x) dx \right)^{1/q} \left(\int_{B(e,t)} v(x) dx \right)^{1/p'} < \infty.$$

Proof. Applying Proposition C, S_α is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$ if and only if

$$\left(\int_G \left(\int_{B(e,r(x))} (S_\alpha^* f)(y) dy \right)^{p'} W^{-p'}(r(x)) w(x) dx \right)^{1/p'} \leq c \left(\int_G g^{q'}(x) v^{1-q'}(x) dx \right)^{1/q'},$$

where

$$(S_\alpha^* f)(x) = \int_{B(e,r(x)/(2c_0))} f(y) r^{\alpha-Q}(xy^{-1}) dy.$$

Now we show that

$$c_1 r^\alpha(x) \int_{B(e,r(x)/(4c_0))} g(s) ds \leq \int_{B(e,r(x))} (S_\alpha^* g)(y) dy \leq c_2 r^\alpha(x) \int_{B(e,r(x)/(2c_0))} g(s) ds, \quad g \geq 0. \quad (3.7)$$

To prove the right-hand side estimate in (3.7) observe that by Tonelli's theorem and Lemma 3.1 we have that

$$\begin{aligned} \int_{B(e,r(x))} (S_\alpha^* g)(y) dy &= \int_{B(e,r(x)/(2c_0))} f(s) \left(\int_{B(e,r(x)) \setminus B(e,2c_0 r(s))} r^{\alpha-Q}(sy^{-1}) dy \right) ds \\ &\leq c_2 r^\alpha(x) \int_{B(e,r(x)/(2c_0))} f(s) ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{B(e,r(x))} (S_\alpha^* g)(y) dy &\geq c r^{\alpha-Q}(x) \left(\int_{B(e,r(x)) \setminus B(e,r(x)/2)} \left(\int_{B(e,r(y)/(2c_0))} f(s) ds \right) dy \right) \\ &\geq c_1 r^\alpha(x) \left(\int_{B(e,r(x)/(4c_0))} f(s) ds \right). \end{aligned}$$

Thus, Theorem A completes the proof. \square

Proof of Theorem 3.1. By (3.5) it is enough to estimate the terms with $J_\alpha f$ and $S_\alpha f$. By applying Proposition 3.1 and Theorem B we have that J_α is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$ if and only if the conditions (ii) and (iii) are satisfied. Now by Lemma 3.2 and the equality (which is a consequence of Proposition A)

$$\left(\int_{G \setminus B(e,t)} W(r(x)) w(x) dx \right)^{1/p'} = \left(\int_{B(e,t)} w(x) dx \right)^{-1/p}$$

we have that S_α is bounded from $L_{dec,r}^p(w, G)$ to $L^q(v, G)$ if and only if (i) is satisfied. \square

4. Multiple Potentials on $G_1 \times G_2$

Let us now investigate the two-weight problem for the operator I_{α,α_2} on the cone $\mathcal{DR}(G_1 \times G_2)$. In the sequel without loss of generality we denote the triangle inequality constants for G_1 and G_2 by one and the same symbol c_0 .

The following statement can be derived just in the same way as Theorem 3.1 was obtained in [4]. The proof is omitted because to avoid repeating those arguments.

Proposition D. *Let $1 < p < \infty$ and let $w(x, y) = w_1(x)w_2(y)$ be a product weight on $G_1 \times G_2$. Then the following relation*

$$\sup_{0 \leq f \downarrow r} \frac{\iint_{G_1 \times G_2} f(x, y) g(x, y) dx dy}{\left(\iint_{G_1 \times G_2} f^p(x, y) w(x, y) \right)^{1/p}} \approx \sum_{i=1}^4 I_k,$$

holds for a non-negative measurable function g , where

$$\begin{aligned}
I_1 &:= \|w\|_{L^1(G_1 \times G_2)}^{-1/p} \|g\|_{L^1(G_1 \times G_2)}, \\
I_2 &:= \|w_2\|_{L^1(G_1)}^{-1/p} \left(\int_{G_1} \int_{B(e_1, r_1(x))} \|g(t, \cdot)\|_{L^1(G_2)} dt \right)^{p'} W_1^{-p'}(r_1(x)) w_1(x) dx \Big)^{1/p'}, \\
I_2 &:= \|w_1\|_{L^1(G_1)}^{-1/p} \left(\int_{G_2} \int_{B(e_2, r_2(y))} \|g(\cdot, \tau)\|_{L^1(G_1)} d\tau \right)^{p'} W_2^{-p'}(r_2(y)) w_2(y) dy \Big)^{1/p'}, \\
I_4 &:= \left(\int_{G_1 \times G_2} \left(\int_{G_1 \times G_2} g(t, \tau) dt d\tau \right)^{p'} W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'}.
\end{aligned}$$

Applying Proposition D together with the duality arguments we can get the following statement (cf. [4]).

Proposition E. *Let $1 < p < \infty$ and let v and w be weights on $G_1 \times G_2$ such that $w(x, y) = w_1(x)w_2(y)$, $\|w\|_{L^1(G_1 \times G_2)} = \infty$. Then an integral operator T defined for functions from $\mathcal{DR}(G_1 \times G_2)$ is bounded from $L_{dec, r}^p(w, G_1 \times G_2)$ to $L^p(v, G_1 \times G_2)$ if and only if for all non-negative measurable g on $G_1 \times G_2$,*

$$\left(\iint_{G_1 \times G_2} \left(\iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} (T^*g)(t, \tau) dt d\tau \right)^{p'} W^{-p'}(x, y) w(x, y) dx dy \right)^{1/p'} \leq C \left(\iint_{G_1 \times G_2} g^q(x, y) v^{1-q'}(x, y) dx dy \right)^{1/q'}.$$

The next statements deals with the double Hardy-type operators defined on $G_1 \times G_2$

$$\begin{aligned}
(H^{a,b}f)(x, y) &= \int_{B(e_1, ar_1(x))} \int_{B(e_2, br_2(x))} f(t, \tau) dt d\tau, \quad (x, y) \in G_1 \times G_2, \\
(\tilde{H}^{a,b}f)(x, y) &= \int_{G_1 \setminus B(e_1, ar_1(x))} \int_{G_2 \setminus B(e_2, br_2(x))} f(t, \tau) dt d\tau, \quad (x, y) \in G_1 \times G_2, \\
(H_1^{a,b}f)(x, y) &= \int_{B(e_1, ar_1(x))} \int_{G_2 \setminus B(e_2, br_2(y))} f(t, \tau) dt d\tau, \quad (x, y) \in G_1 \times G_2, \\
(H_2^{a,b}f)(x, y) &= \int_{G_1 \setminus B(e_1, ar_1(x))} \int_{B(e_2, br_2(y))} f(t, \tau) dt d\tau, \quad (x, y) \in G_1 \times G_2.
\end{aligned}$$

Proposition 4.1. *Let $1 < p \leq q < \infty$. Suppose that v and w be weights on $G_1 \times G_2$ such that either $w(x, y) = w_1(x)w_2(y)$ or $v(x, y) = v_1(x)v_2(y)$. Then*

(i) The operator $H^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if

$$A := \sup_{t>0, \tau>0} \left(\int_{G_1 \setminus B(e_1, t)} \int_{G_2 \setminus B(e_2, \tau)} v(x, y) dx dy \right)^{1/q} \left(\int_{B(e_1, at)} \int_{B(e_2, b\tau)} w^{1-p'}(x, y) dx dy \right)^{1/p'} < \infty.$$

(ii) The operator $\tilde{H}^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if

$$\sup_{t>0, \tau>0} \left(\int_{B(e_1, t)} \int_{B(e_2, \tau)} v(x, y) dx dy \right)^{1/q} \left(\int_{G_1 \setminus B(e_1, at)} \int_{G_2 \setminus B(e_2, b\tau)} w^{1-p'}(x, y) dx dy \right)^{1/p'} < \infty.$$

(iii) The operator $H_1^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if

$$\sup_{t>0, \tau>0} \left(\int_{G_1 \setminus B(e_1, t)} \int_{B(e_2, \tau)} v(x, y) dx dy \right)^{1/q} \left(\int_{B(e_1, at)} \int_{G_2 \setminus B(e_2, b\tau)} w^{1-p'}(x, y) dx dy \right)^{1/p'} < \infty.$$

(iv) The operator $H_2^{a,b}$ is bounded from $L^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if

$$\sup_{t>0, \tau>0} \left(\int_{B(e_1, t)} \int_{G_2 \setminus B(e_2, \tau)} v(x, y) dx dy \right)^{1/q} \left(\int_{G_1 \setminus B(e_1, at)} \int_{B(e_2, b\tau)} w^{1-p'}(x, y) dx dy \right)^{1/p'} < \infty.$$

Proof. Let $w(x, y) = w_1(x)w_2(y)$. Then the proposition follows in the same way as the appropriate statements regarding the Hardy operators defined on \mathbb{R}_+^2 in [14], [12] (see also Theorem 1.1.6 of [13]). If v is a product weight, i.e. $v(x, y) = v_1(x)v_2(y)$, then the result follows from the duality arguments. We give the proof, for example, for $H^{a,b}$ in the case when $w(x, y) = w_1(x)w_2(y)$.

First suppose that $S := \int_{G_2} w_2^{1-p'}(y) dy = \infty$. Let $\{x_k\}_{k=-\infty}^{+\infty}$ be a sequence of positive numbers for which the equality

$$2^k = \int_{B(e_2, bx_k)} w_2^{1-p'}(y) dy \quad (4.1)$$

holds for all $k \in \mathbb{Z}$. This equality follows because of the continuity in t of the integral over the ball $B(e_2, bt)$. It is clear that $\{x_k\}$ is increasing and $\mathbb{R}_+ = \cup_{k \in \mathbb{Z}} [x_k, x_{k+1})$. Moreover, it is easy to verify that

$$2^k = \int_{B(e_2, bx_{k+1}) \setminus B(e_2, bx_k)} w_2^{1-p'}(y) dy.$$

Let $f \geq 0$. We have that

$$\begin{aligned} \|H^{a,b} f\|_{L_v^q(G_1 \times G_2)}^q &= \iint_{G_1 \times G_2} v(x, y) (H^{a,b} f)^q(x, y) dx dy \\ &\leq \sum_{k \in \mathbb{Z}} \int_{G_1} \int_{B(e_2, x_{k+1}) \setminus B(e_2, x_k)} v(x, y) \left(\iint_{B(e_1, ar_1(x)) \times B(e_2, br_2(x))} f(t, \tau) dt d\tau \right)^q dx dy \\ &\leq \sum_{k \in \mathbb{Z}} \int_{G_1} \left(\int_{B(e_2, x_{k+1}) \setminus B(e_2, x_k)} v(x, y) dy \right) \left(\int_{B(e_1, ar_1(x))} \left(\int_{B(e_2, bx_{k+1})} f(t, \tau) d\tau \right) dt \right)^q dx \\ &= \sum_{k \in \mathbb{Z}} \int_{G_1} V_k(x) \left(\int_{B(e_1, ar_1(x))} F_k(t) dt \right)^q dx, \end{aligned}$$

where

$$V_k(x) := \int_{B(e_2, x_{k+1}) \setminus B(e_2, x_k)} v(x, y) dy; \quad F_k(t) := \int_{B(e_2, bx_{k+1})} f(t, \tau) d\tau.$$

It is obvious that

$$A^q \geq \sup_{\substack{a>0 \\ j \in \mathbb{Z}}} \left(\int_{G_1 \setminus B(e_1, t)} v_j(y) dy \right) \left(\iint_{B(e_1, at) \times B(e_2, bx_j)} w^{1-p'}(x, y) dx dy \right)^{q/p'}.$$

Hence, by Theorem A

$$\begin{aligned} \|H^{a,b} f\|_{L_v^q(G_1 \times G_2)}^q &\leq c A^q \sum_{j \in \mathbb{Z}} \left[\int_{G_1} w_1(x) \left(\int_{B(e_2, bx_j)} w_2^{1-p'}(y) dy \right)^{1-p} (F_k(x))^p dx \right]^{q/p} \\ &\leq c A^q \left[\int_{G_1} w_1(x) \sum_{j \in \mathbb{Z}} \left(\int_{B(e_2, bx_j)} w_2^{1-p'}(y) dy \right)^{1-p} \left(\sum_{k=-\infty}^j \int_{B(e_2, bx_{k+1}) \setminus B(e_2, bx_k)} f(x, \tau) d\tau \right)^p dx \right]^{q/p}. \end{aligned}$$

On the other hand, (4.1) yields that

$$\begin{aligned} &\sum_{k=n}^{+\infty} \left(\int_{B(e_2, bx_k)} w_2^{1-p'}(y) dy \right)^{1-p} \left(\sum_{k=-\infty}^n \int_{B(e_2, bx_{k+1}) \setminus B(e_2, bx_k)} w_2^{1-p'}(y) dy \right)^{p-1} \\ &= \sum_{k=n}^{+\infty} \left(\int_{B(e_2, bx_k)} w_2^{1-p'}(y) dy \right)^{1-p} \left(\int_{B(e_2, bx_{n+1})} w_2^{1-p'}(y) dy \right)^{p-1} = \left(\sum_{k=n}^{+\infty} 2^{k(1-p)} \right) 2^{(n+1)(p-1)} \leq c \end{aligned}$$

for all $n \in \mathbb{Z}$. Hence by the discrete Hardy inequality (see e.g. [6]) and Hölder's inequality we have

$$\begin{aligned} \|H^{a,b}f\|_{L_v^q(G_1 \times G_2)}^q &\leq cA^q \left[\int_{G_1} w_1(x) \sum_{j \in \mathbb{Z}} \left(\int_{B(e_2, bx_{j+1}) \setminus B(e_2, bx_j)} w_2^{1-p'}(y) dy \right)^{1-p} \left(\int_{B(e_2, bx_{j+1}) \setminus B(e_2, bx_j)} f(x, \tau) d\tau \right)^p dx \right]^{q/p} \\ &\leq cA^q \left[\int_{G_1} w_1(x) \sum_{j \in \mathbb{Z}} \left(\int_{B(e_2, bx_{j+1}) \setminus B(e_2, bx_j)} w_2(\tau) f^p(x, \tau) d\tau \right) dx \right]^{q/p} = cA^q \|f\|_{L_w^p(G_1 \times G_2)}^q. \end{aligned}$$

If $S < \infty$, then without loss of generality we can assume that $S = 1$. In this case we choose the sequence $\{x_k\}_{k=-\infty}^0$ for which (4.1) holds for all $k \in \mathbb{Z}_-$. Arguing as in the case $S = \infty$ and using slight modification of the discrete Hardy inequality (see also [13], Chapter 1 for similar arguments), we finally obtain the desired result.

Finally we notice that the part (i) can be also proved if we first establish the boundedness of the operator $(\mathcal{H}^{a,b}\varphi)(t, \tau) = \int_0^{at+b\tau} \varphi(s, r) ds dr$ in the spirit of Theorem 1.1.6 in [13] and then pass to the case of $G_1 \times G_2$ by Proposition A. \square

The next statement will be useful for us.

Proposition 4.2. *Let $1 < p \leq q < \infty$. Assume that v and w are weights on $G_1 \times G_2$. Suppose that $w(x, y) = w_1(x)w_2(y)$ and that $W_i(\infty) = \infty$, $i = 1, 2$. Then the operator $H^{1,1}$ is bounded from $L_{dec,r}^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if the following four conditions are satisfied:*

(i)

$$\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} w(x, y) dx dy \right)^{-1/p} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{Q_1 q}(x) r_2(y) Q_2 q v(x, y) dx dy \right)^{1/q} < \infty;$$

(ii)

$$\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{Q_1 p'}(x) r_2(y) Q_2 p' W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'} \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} v(x, y) dx dy \right)^{1/q} < \infty;$$

(iii)

$$\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} w_1(r_1(x)) dx \right)^{-1/p} \left(\int_{B(e_2, a_2)} r_2(y) Q_2 p' W_2^{-p'}(r_2(y)) w_2(y) dy \right)^{1/p'} \left(\int_{B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} r_1(x) Q_1 q v(x, y) dx dy \right)^{1/q} < \infty;$$

(iv)

$$\sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_1(x) Q_1 p' W_1^{-p'}(r_1(x)) w_1(x) dt_1 \right)^{1/p'} \left(\int_{B(e_2, a_2)} w_2(y) dy \right)^{-1/p} \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{B(e_2, a_2)} r_2(y) Q_2 q v(x, y) dx dy \right)^{1/q} < \infty.$$

Proof. We follow the proof of Theorem 5.3 in [4]. First of all observe that by Proposition E, if w is a product weight, i.e., $w(x_1, x_2) = w_1(x_1)w_2(x_2)$, such that $W_i(\infty) = \infty$, $i = 1, 2$, and v is any weight on $G_1 \times G_2$, then $H^{1,1}$ is bounded from $L_{dec,r}^p(w, G_1)$ to $L^q(v, G_2)$ if and only if

$$\begin{aligned} &\left(\iiint_{G_1 \times G_2} \left(\int_{B(e_1, r_1(x))} \int_{B(e_2, r_2(x))} \left[\int_{G_1 \setminus B(e_1, r_1(t))} \int_{G_1 \setminus B(e_2, r_2(\tau))} g(s, \varepsilon) ds d\varepsilon \right] dt d\tau \right)^{p'} W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'} \\ &\leq c \left(\iint_{G_1 \times G_2} g^{q'}(x, y) v^{1-q'}(x, y) dx dy \right)^{1/q'}, \quad g \geq 0. \end{aligned} \quad (4.2)$$

Further, we have that

$$\begin{aligned}
& \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(x))} \left(\int_{G_1 \setminus B(e_1, r_1(t))} \int_{G_2 \setminus B(e_2, r_2(t))} g(s, \varepsilon) ds d\varepsilon \right) dt d\tau \\
&= \int_{B(e_1, r_1(x))} \int_{B(e_2, r_2(x))} r_1^{Q_1}(t) r_2^{Q_2}(\tau) g(t, \tau) dt d\tau + r_1^{Q_1}(x) \int_{G_1 \setminus B(e_1, r_1(x))} \int_{B(e_2, r_2(y))} r_2^{Q_2}(\tau) g(t, \tau) dt d\tau \\
&+ r_2^{Q_2}(y) \int_{B(e_1, r_1(x))} \int_{G_2 \setminus B(e_2, r_2(y))} r_1^{Q_1}(t) g(t, \tau) dt d\tau \\
&+ r_1^{Q_1}(x) r_2^{Q_2}(y) \int_{G_1 \setminus B(e_1, r_1(x))} \int_{G_2 \setminus B(e_2, r_2(y))} g(t, \tau) dt d\tau \\
&=: I^{(1)}(x, y) + I^{(2)}(x, y) + I^{(3)}(x, y) + I^{(4)}(x, y).
\end{aligned}$$

It is obvious that (4.2) holds if and only if

$$\left(\iint_{G_1 \times G_2} (I^{(j)})^{p'}(x, y) W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'} \leq c \left(\iint_{G_1 \times G_2} g^{q'}(x, y) v^{1-q'}(x, y) dx dy \right)^{1/q'} \quad (4.3)$$

for $j = 1, 2, 3, 4$. By using Proposition 4.1 (Part (i)) we find that

$$\left(\iint_{G_1 \times G_2} (I^{(1)})^{p'}(x, y) W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'} \leq c \left(\iint_{G_1 \times G_2} g^{q'}(x, y) v^{1-q'}(x, y) dx dy \right)^{1/q'}$$

if and only if

$$\begin{aligned}
& \left(\int_{G_1 \setminus B(e_1, t)} \int_{G_2 \setminus B(e_2, \tau)} W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'} \left(\iint_{B(e_1, t) \times B(e_2, \tau)} \left(\frac{v^{1-q'}(x, y)}{r_1^{Q_1, q'}(x) r_2^{Q_2, q'}(y)} \right)^{1-q} dx dy \right)^{1/q} \\
&= c_p \left(\iint_{B(e_1, t) \times B(e_2, \tau)} w(x, y) dx dy \right)^{-1/p} \left(\iint_{B(e_1, t) \times B(e_2, \tau)} v(x, y) r_1^{Q_1 q}(x) r_2^{Q_2 q}(y) dx dy \right)^{1/q} \leq C.
\end{aligned}$$

In the latter equality we used the equality

$$\left(\int_{G_i \setminus B(e_i, t)} W_i^{-p'}(r_i(x)) w_i(x) dx \right)^{1/p'} = \left(\int_{B(e_i, t)} w_i(x) dx \right)^{-1/p}, \quad i = 1, 2,$$

which is direct consequence of integration by parts and Proposition A. Taking now Proposition 4.1 (Part (ii)) into account we find that (4.3) holds for $j = 4$ if and only if condition (ii) is satisfied, while Proposition 4.1 (Parts (iii) and (iv)) and the following observation:

$$\begin{aligned}
& \sup_{a_1, a_2 > 0} \left(\int_{G_1 \setminus B(e_1, a_1)} w_1(x) W_1^{-p'}(r_1(x)) dx \right)^{1/p'} \left(\int_{B(e_2, a_2)} r_2^{p' Q_2}(y) W_2^{-p'}(r_2(y)) w_2(y) dy \right)^{1/p'} \\
&\times \left(\int_{B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} r_1^{Q_1 q}(x) v(x, y) dx dy \right)^{1/q} \\
&= c_p \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} w_1(x) dx \right)^{-1/p} \left(\int_{B(e_2, a_2)} r_2^{Q_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) dy \right)^{1/p'} \\
&\times \left(\int_{B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} r_1^{Q_1 q}(x) v(x, y) dx dy \right)^{1/q} < \infty;
\end{aligned}$$

$$\begin{aligned}
& \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) dx \right)^{1/p'} \left(\int_{G_2 \setminus B(e_2, a_2)} w_2(y) W_2^{-p'}(r_2(y)) dy \right)^{1/p'} \\
& \times \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{B(e_2, a_2)} r_2^{Q_2 q}(y) v(x, y) dx dy \right)^{1/q} \\
& = c_p \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) dx \right)^{1/p'} \left(\int_{B(e_2, a_2)} w_2(t_2) dt_2 \right)^{-1/p} \\
& \times \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{B(e_2, a_2)} r_2^{Q_2 q}(y) v(x, y) dx dy \right)^{1/q} < \infty
\end{aligned}$$

yield (4.3) for $j = 2, 3$. □

Let

$$\begin{aligned}
(J_{\alpha_1, \alpha_2} f)(x, y) &= \int_{B(e_1, 2c_0 r_1(x))} \int_{B(e_2, 2c_0 r_2(y))} f(t, \tau) r_1(xt^{-1})^{\alpha_1 - Q_1} r_2(y\tau^{-1})^{\alpha_2 - Q_2} dt d\tau, \\
(J_{\alpha_1} S_{\alpha_2} f)(x, y) &= \int_{B(e_1, 2c_0 r_1(x))} \int_{G_2 \setminus B(e_2, 2c_0 r_2(y))} f(t, \tau) r_1(xt^{-1})^{\alpha_1 - Q_1} r_2(y\tau^{-1})^{\alpha_2 - Q_2} dt d\tau, \\
(S_{\alpha_1} J_{\alpha_2} f)(x, y) &= \int_{G_1 \setminus B(e_1, 2c_0 r_1(x))} \int_{B(e_2, 2c_0 r_2(y))} f(t, \tau) r_1(xt^{-1})^{\alpha_1 - Q_1} r_2(y\tau^{-1})^{\alpha_2 - Q_2} dt d\tau, \\
(S_{\alpha_1, \alpha_2} f)(x, y) &= \int_{G_1 \setminus B(e_1, 2c_0 r_1(x))} \int_{G_2 \setminus B(e_2, 2c_0 r_2(y))} f(t, \tau) r_1(xt^{-1})^{\alpha_1 - Q_1} r_2(y\tau^{-1})^{\alpha_2 - Q_2} dt d\tau,
\end{aligned}$$

where c_0 is the constant from the triangle inequality for the homogeneous norms r_1 and r_2 .

It is obvious that

$$I_{\alpha_1, \alpha_2} f = J_{\alpha_1, \alpha_2} f + J_{\alpha_1} S_{\alpha_2} f + S_{\alpha_1} J_{\alpha_2} f + S_{\alpha_1, \alpha_2} f. \quad (4.4)$$

Now we formulate the main result of this section.

Theorem 4.1. *Let $1 < p \leq q < \infty$. Assume that v and w are weights on $G_1 \times G_2$ such that $w(x, y) = w_1(x)w_2(y)$. Suppose that either $w_i \in DC^{\alpha_i, p}$, $i = 1, 2$, or $v \in DC(x) \cap DC(y)$. Then the operator I_{α_1, α_2} is bounded from $L_{dec, r}^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if the following conditions are satisfied:*

(i)

$$A_1 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} w(x, y) dx dy \right)^{-1/p} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} (r_1^{\alpha_1}(x) r_2^{\alpha_2}(y))^q v(x, y) dx dy \right)^{1/q} < \infty;$$

(ii)

$$\begin{aligned}
A_2 &:= \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{Q_1 p'}(x) r_2^{Q_2 p'}(y) W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'} \\
&\times \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} (r_1^{\alpha_1 - Q_1}(x) r_2^{\alpha_2 - Q_2}(y))^q v(x, y) dx dy \right)^{1/q} < \infty;
\end{aligned}$$

(iii)

$$A_3 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} w_1(x) dx \right)^{-1/p} \left(\int_{B(e_2, a_2)} r_2^{Q_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) dy \right)^{1/p'}$$

$$\times \left(\int_{B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} r_1^{\alpha_1 q}(x) r_2^{q(\alpha_2 - Q_2)}(y) v(x, y) dx dy \right)^{1/q} < \infty;$$

(iv)

$$A_4 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) dx \right)^{1/p'} \left(\int_{B(e_2, a_2)} w_2(y) dy \right)^{-1/p} \\ \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{q(\alpha_1 - Q_1)}(x) r_2^{q\alpha_2}(y) v(x, y) dx dy \right)^{1/q} < \infty.$$

(v)

$$A_5 := \sup_{a_1, a_2 > 0} \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} r_1^{\alpha_1 p'}(x) r_2^{\alpha_2 p'}(y) W^{-p'}(r_1(x), r_2(y)) w(x, y) dx dy \right)^{1/p'} \\ \times \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} v(x, y) dx dy \right)^{1/q} < \infty;$$

(vi)

$$A_6 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} w_1(x) dx \right)^{-1/p} \left(\int_{G_2 \setminus B(e_2, a_2)} r_2^{\alpha_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) dy \right)^{1/p'} \\ \times \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{\alpha_1 q}(x) v(x, y) dx dy \right)^{1/q} < \infty;$$

(vii)

$$A_7 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) dx \right)^{1/p'} \left(\int_{G_2 \setminus B(e_2, a_2)} r_2^{\alpha_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) dy \right)^{1/p'} \\ \times \left(\int_{G_1 \setminus B(e_1, a_1)} \int_{B(e_2, a_2)} r_1^{(\alpha_1 - Q_1)q}(x) v(x, y) dx dy \right)^{1/q} < \infty;$$

(viii)

$$A_8 := \sup_{a_1, a_2 > 0} \left(\int_{G_2 \setminus B(e_1, a_1)} r_1^{\alpha_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) dx \right)^{-1/p} \left(\int_{B(e_2, a_2)} w_2(y) dy \right)^{1/p'} \\ \times \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} r_2^{\alpha_2 q}(x) v(x, y) dx dy \right)^{1/q} < \infty;$$

(ix)

$$A_9 := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} r_2^{Q_2 p'}(y) W_2^{-p'}(r_2(y)) w_2(y) dy \right)^{1/p'} \left(\int_{G_1 \setminus B(e_1, a_1)} r_1^{\alpha_1 p'}(x) W_1^{-p'}(r_1(x)) w_1(x) dx \right)^{1/p'} \\ \times \left(\int_{B(e_1, a_1)} \int_{G_2 \setminus B(e_2, a_2)} r_2^{(\alpha_2 - Q_2)q}(y) v(x, y) dx dy \right)^{1/q} < \infty.$$

Proof. Let us assume that $v \in DC(x) \cap DC(y)$. The case when $w_i \in DC^{\alpha_i, p}(G_i)$, $i = 1, 2$ follows analogously. By using representation (4.4) we have to investigate the boundedness of the operators $J_{\alpha_1, \alpha_2} f$, $J_{\alpha_1} S_{\alpha_2} f$, $S_{\alpha_1} J_{\alpha_2} f$, $S_{\alpha_1, \alpha_2} f$ separately.

Since $f \in \mathcal{DR}(G_1 \times G_2)$ by using the arguments of the proof of Proposition 3.1 it can be checked that

$$(J_{\alpha_1, \alpha_2} f)(x, y) \approx r_1^{\alpha_1 - Q_1}(x) r_2^{\alpha_2 - Q_2}(y) \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} f(t, \tau) dt d\tau$$

(see also [16] for similar estimate in the case of the multiple one-sided potentials on \mathbb{R}_+^2). Hence, by Proposition 4.2 we have that J_{α_1, α_2} is bounded from $L_{dec, r}^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if conditions (i)- (iv) hold.

Observe that the dual to S_{α_1, α_2} is given by

$$(S_{\alpha_1, \alpha_2}^* g)(x, y) = \iint_{B(e_1, r_1(x)/(2c_0)) \times B(e_2, r_2(y)/(2c_0))} g(t, \tau) r_1^{\alpha_1 - Q_1}(xt^{-1}) r_2^{\alpha_2 - Q_2}(y\tau^{-1}) dt d\tau.$$

Further, Tonelli's theorem together with Lemma 3.1 for both variables implies that there are positive constants c_1 and c_2 such that for all $(x, y) \in G_1 \times G_2$ for the dual (see also the proof of Lemma 3.2)

$$\begin{aligned} & r_1^{\alpha_1}(x) r_2^{\alpha_2}(y) \iint_{B(e_1, r_1(x)/(4c_0)) \times B(e_2, r_2(y)/(4c_0))} g(t, \tau) dt d\tau \leq c_1 \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} (S_{\alpha_1, \alpha_2}^* g)(t, \tau) dt d\tau \\ & \leq c_2 r_1^{\alpha_1}(x) r_2^{\alpha_2}(y) \iint_{B(e_1, r_1(x)/(2c_0)) \times B(e_2, r_2(y)/(2c_0))} g(t, \tau) dt d\tau. \end{aligned}$$

Applying Propositions 4.1 and 4.2 with the condition that $v \in DC(G_1 \times G_2)$ we find that the operator S_{α_1, α_2} is bounded from $L_{dec, r}^p(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if condition (v) is satisfied.

Further, observe that due to the fact that f is radially decreasing with respect to the first variable we have

$$(J_{\alpha_1} S_{\alpha_2} f)(x, y) \approx (\mathcal{H}_{\alpha_1} S_{\alpha_2} f)(x, y),$$

where

$$(\mathcal{H}_{\alpha_1} S_{\alpha_2} f)(x, y) = r_1^{\alpha_1 - Q_1}(x) \int_{B(e_1, 2c_0 r_1(x))} \int_{G_2 \setminus B(e_2, 2c_0 r_2(y))} f(t, \tau) r_2(y\tau^{-1})^{\alpha_2 - Q_2} dt d\tau.$$

Dual of $\mathcal{H}_{\alpha_1} S_{\alpha_2}$ is given by

$$(\mathcal{H}_{\alpha_1}^* S_{\alpha_2}^* g)(t, \tau) = \int_{G_1 \setminus B(e_1, r(t))} \int_{B(e_2, r(\tau)/(2c_0))} r_1^{\alpha_1 - Q_1}(s) r_2^{\alpha_2 - Q_2}(\tau\epsilon^{-1}) f(s, \epsilon) ds d\epsilon.$$

Further, we have

$$\begin{aligned} T(x, y) &:= \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} (\mathcal{H}_{\alpha_1}^* S_{\alpha_2}^* g)(t, \tau) dt d\tau \\ &= \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} \left(\int_{B(e_1, r_1(x)) \setminus B(e_1, r(t))} \int_{B(e_2, r(\tau)/(2c_0))} r_1^{\alpha_1 - Q_1}(s) r_2^{\alpha_2 - Q_2}(\tau\epsilon^{-1}) f(s, \epsilon) ds d\epsilon \right) dt d\tau \\ &+ \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} \left(\int_{G_1 \setminus B(e_1, r_1(x))} \int_{B(e_2, r(\tau)/(2c_0))} r_1^{\alpha_1 - Q_1}(s) r_2^{\alpha_2 - Q_2}(\tau\epsilon^{-1}) f(s, \epsilon) ds d\epsilon \right) dt d\tau \\ &=: T_1(x, y) + T_2(x, y). \end{aligned}$$

Tonelli's theorem for G_1 , the inequality $r_2^{\alpha_2 - Q_2}(\tau\epsilon^{-1}) \geq c r_2^{\alpha_2 - Q_2}(y)$ for $\tau \in B(e_2, r(y))$, $\epsilon \in B(e_2, r(\tau)/(2c_0))$, and the

fact that the integral $\int_{B(e_1, \tau)} f(s, \varepsilon) ds$ is decreasing in τ uniformly to ε yield that

$$\begin{aligned}
T_1(x, y) &\geq cr_2^{\alpha_2 - Q_2}(y) \int_{B(e_1, r_1(x))} \int_{B(e_2, r_2(y)) \setminus B(e_2, r_2(y)/2)} \left(\int_{B(e_1, r_1(x)) \setminus B(e_1, r(t))} \int_{B(e_2, r_2(y)/(4c_0))} r_1^{\alpha_1 - Q_1}(s) f(s, \varepsilon) ds d\varepsilon \right) dt d\tau \\
&= cr_2^{\alpha_2}(y) \int_{B(e_1, r_1(x))} \left(\int_{B(e_1, r_1(x)) \setminus B(e_1, r(t))} \left(r_1^{\alpha_1 - Q_1}(s) \left(\int_{B(e_2, r_2(y)/(4c_0))} f(s, \varepsilon) d\varepsilon \right) ds \right) dt \right) \\
&= cr_2^{\alpha_2}(y) \int_{B(e_1, r_1(x))} \left(\int_{B(e_1, r_1(x)) \setminus B(e_1, r(t))} F(s, y) ds \right) dt = cr_2^{\alpha_2}(y) \int_{B(e_1, r_1(x))} F(s, y) \left(\int_{B(e_1, r(s))} dt \right) ds \\
&= cr_2^{\alpha_2}(y) \int_{B(e_1, r_1(x))} \int_{B(e_2, r_2(y)/(4c_0))} r_1^{\alpha_1}(s) f(t, \tau) d\varepsilon ds.
\end{aligned}$$

Here we used the notation

$$F(s, y) := \int_{B(e_2, r_2(y)/(4c_0))} f(s, \varepsilon) d\varepsilon.$$

Taking into account that the function $\int_{B(e_2, 2c_0\lambda)} f(s, \varepsilon) d\varepsilon$ is decreasing in λ uniformly to s , the inequality $r_2(\tau\varepsilon^{-1}) \leq cr_2(y)$ for $\tau \in B(e_2, r(y))$, $\varepsilon \in B(e_2, r(\tau)/(2c_0))$, and Tonelli's theorem for G_1 we find that

$$T_2(x, y) \geq cr_1^{Q_1}(x) r_2^{\alpha_2}(y) \int_{G_1 \setminus B(e_1, r_1(x))} \int_{B(e_2, r_2(y)/(4c_0))} r_1^{\alpha_1 - Q_1}(s) f(t, \tau) d\varepsilon ds.$$

To get the upper estimate, observe that Tonelli's theorem for $G_1 \times G_2$ and Lemma 3.1 for r_2 yield that

$$\begin{aligned}
T_1(x, y) &\leq \int_{B(e_1, r_1(x))} \int_{B(e_2, r_2(y)/(2c_0))} r_1^{\alpha_1 - Q_1}(s) f(s, \varepsilon) \left(\int_{B(e_1, r_1(s))} \int_{B(e_2, r_2(y)) \setminus B(e_2, 2c_0 r_2(\varepsilon))} r_2^{\alpha_2 - Q_2}(\tau\varepsilon^{-1}) dt d\tau \right) ds d\varepsilon \\
&\leq cr_2^{\alpha_2}(y) \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y)/(2c_0))} r_1^{\alpha_1}(s) f(s, \varepsilon) ds d\varepsilon.
\end{aligned}$$

Similarly,

$$T_2(x, y) \leq cr_1^{Q_1}(x) r_2^{\alpha_2}(y) \iint_{G_1 \setminus B(e_1, r_1(x)) \times B(e_2, r_2(y)/(2c_0))} r_1^{\alpha_1 - Q_1}(s) f(s, \varepsilon) ds d\varepsilon.$$

Summarizing these estimates we see that there are positive constants c_1 and c_2 depending only on α_1 , α_2 , Q_1 and Q_2 such that

$$\begin{aligned}
&r_2^{\alpha_2}(y) \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y)/(4c_0))} r_1^{\alpha_1}(s) f(s, \varepsilon) ds d\varepsilon \\
&+ r_1^{Q_1}(x) r_2^{\alpha_2}(y) \iint_{G_1 \setminus B(e_1, r_1(x)) \times B(e_2, r_1(y)/(4c_0))} r_1^{\alpha_1 - Q_1}(s) f(s, \varepsilon) ds d\varepsilon \\
&\leq c_1 T(x, y) \leq r_2^{\alpha_2}(y) \iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y)/(2c_0))} r_1^{\alpha_1}(s) f(s, \varepsilon) ds d\varepsilon \\
&+ r_1^{Q_1}(x) r_2^{\alpha_2}(y) \iint_{G_1 \setminus B(e_1, r_1(x)) \times B(e_2, r_1(y)/(2c_0))} r_1^{\alpha_1 - Q_1}(s) f(s, \varepsilon) ds d\varepsilon.
\end{aligned}$$

Taking Propositions 4.1 and E into account together with the doubling condition for v with respect to the second variable we see that the operator $J_{\alpha_1} S_{\alpha_2}$ is bounded from $L^p_{dec,r}(w, G_1)$ to $L^q(v, G_2)$ if and only if the conditions (vi) and (vii) are satisfied.

By the similar manner (changing the roles of the first and second variables) we can get that $S_{\alpha_1} J_{\alpha_2}$ is bounded from $L^p_{dec,r}(w, G_1)$ to $L^q(v, G_2)$ if and only if the conditions (viii) and (ix) are satisfied. \square

Theorem 4.1 and Remark 2.1 imply criteria for the trace inequality for I_{α_1, α_2} . Namely the following statement holds:

Theorem 4.2. *Let $1 < p \leq q < \infty$ and let $0 < \alpha_i < Q_i/p$, $i = 1, 2$. Then I_{α_1, α_2} is bounded from $L^p_{dec,r}(G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if the following condition holds*

$$B := \sup_{a_1, a_2 > 0} \left(\int_{B(e_1, a_1)} \int_{B(e_2, a_2)} v(x, y) dx dy \right)^{1/q} a_1^{\alpha_1 - Q_1/p} a_2^{\alpha_2 - Q_2/p} < \infty.$$

Proof. Sufficiency is a consequence of the inequality $\max\{A_1, \dots, A_9\} \leq cB$, while necessity follows immediately by taking the test function $f_{a_1, a_2}(x, y) = \chi_{B(e_1, a_1)}(x) \chi_{B(e_2, a_2)}(y)$, $a_1, a_2 > 0$. \square

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